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The flow induced by torsional oscillations of infinite planes

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A viscous fluid is confined between two parallel, infinite planes which perform torsional oscillations of small amplitude about a common axis. The resulting flow is studied for the case of high-frequency oscillations, when boundary layers form adjacent to moving surfaces. Particular analysis is made of the second-order, steady, radial-axial streaming. It is shown that in certain circumstances viscosity may be effective throughout the domain of flow, while in others there is a region in which viscosity is negligible.

1. Introduction

The flow induced in a semi-infinite viscous fluid by the torsional oscillations of a plane has been studied by several authors (Rosenblat 1959; Benney 1964; Riley 1965), and its properties are well understood. Its most important characteristics, which it shares with other flows resulting from the oscillations of an immersed solid (see for example Riley 1967, which contains an extensive list of references), are the formation of an oscillatory (Stokes) shear layer at the moving boundary, and the development of a second-order, time-independent streaming. This latter flow may extend throughout the fluid or may, for certain values of the relevant physical parameters, be confined within a second layer in the neighbourhood of the boundary. The existence of two regions of flow necessitates the use of a suitable approximation technique, such as the method of matched asymptotic expansions (cf. Van Dyke 1964), to obtain solutions.

The subject of the present paper is an extension of the above situation, in the sense that there is a second boundary present confining the fluid. More specifically, the fluid is bounded by two parallel planes, both of which perform torsional oscillations about a common axis.

A partial solution to this problem was given by Rosenblat (1960), but only for a narrow range of parameters, which in fact excluded the formation of the second boundary layer mentioned earlier. In the following, this restriction on the parameters is removed, and our interest is focused on the flow in this secondary layer and in the region beyond it.

With respect to a cylindrical polar co-ordinate system (r, θ, z), the fluid is taken to be confined between parallel, rigid planes $z = 0$ and $z = d$, which may oscillate about a common axis $r = 0$. If (u, v, w) are the velocity components in

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this co-ordinate system, torsional oscillations of the plane $z = 0$ (henceforth referred to for convenience as the 'lower' plane), of given frequency σ and angular speed ω, determine one set of boundary conditions, namely,

$$u = w = 0, \quad v = r\omega e^{i\sigma t} \quad \text{on} \quad z = 0, \quad \text{all} \quad r, \theta, t. \tag{1.1}$$

A similar motion of the 'upper' plane $z = d$ fixes analogous conditions there:

$$u = w = 0, \quad v = r\omega_1 e^{i\sigma t} \quad \text{on} \quad z = d, \quad \text{all} \quad r, \theta, t, \tag{1.2}$$

where in general ω_1 may be complex to allow for a possible phase difference.

This most general boundary-value problem has been studied in detail by Jones (1968). In this paper, however, we restrict ourselves to two special cases which are typical in the sense that their solutions include and exemplify the main features of more general situations. These two cases have (1.1) as the lower boundary condition for both, and either

$$u = w = 0, \quad v = 0 \quad \text{on} \quad z = d, \tag{1.3a}$$

or

$$u = w = 0, \quad v = r\omega e^{i\sigma t} \quad \text{on} \quad z = d \tag{1.3b}$$

as the upper boundary condition. Thus, in the former the plane $z = d$ is at rest, while in the latter it has the same motion as the lower plane.

The flows specified by the conditions (1.1) and (1.3) involve four length scales: (i) radial distance r, (ii) axial distance d, (iii) amplitude of oscillation $r\omega/\sigma$, and (iv) Stokes-layer thickness $(\nu/\sigma)^{\frac{1}{2}}$, ν being the kinematic viscosity. From these four quantities it is possible to construct exactly three independent, dimensionless parameters, namely, (i) r^*/d, where r^* is a measure of length in the radial direction, (ii) amplitude parameter ω/σ, which is the ratio of the amplitude of the oscillations to the radial distance, and (iii) frequency parameter $d(\sigma/\nu)^{\frac{1}{2}}$, the ratio of the Stokes-layer thickness to the distance between the planes. In the present problem, however, the plane geometry of the system allows a solution in which r is a similarity variable, so that the parameter r^*/d does not appear in the analysis. It follows that the motion is characterized by the two remaining parameters, which we designate

$$\epsilon = \omega/\sigma \quad \text{and} \quad \lambda = d\sqrt{(\sigma/\nu)}. \tag{1.4}$$

We shall in the following be concerned only with flows in which

$$\epsilon \ll 1 \quad \text{and} \quad \lambda \gg 1, \tag{1.5}$$

that is, small amplitude oscillations at high frequency. It will soon become apparent, however, that the relative magnitudes of ϵ and λ^{-1} are crucial to the nature of the solution.

2. Development of the solution

The flow is governed by the Navier–Stokes and continuity equations which, in the usual notation, are

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{q} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{q} \tag{2.1}$$

and

$$\nabla \cdot \mathbf{q} = 0. \tag{2.2}$$

It is consistent with the boundary conditions (1.1) and (1.3) to seek a solution in which all motions are axi-symmetric, and such that the velocity components and pressure may be non-dimensionalized as follows:

$$u = r\omega(\partial/\partial z')F(z',t'), \quad v = r\omega G(z',t'), \quad w = -2\omega d F(z',t'),$$
$$p/\rho = \omega^2 d^2 P(z',t') + \tfrac{1}{2}\omega^2 r^2 K(t'), \tag{2.3}$$

where
$$z' = z/d \quad \text{and} \quad t' = \sigma t \tag{2.4}$$

are dimensionless distance and time respectively. With these transformations, equation (2.2) is identically satisfied, and equations (2.1) become, on omitting the primes,

$$\frac{\partial^2 F}{\partial z \partial t} + \epsilon\left[\left(\frac{\partial F}{\partial z}\right)^2 - 2F\frac{\partial^2 F}{\partial z^2} - G^2 + K(t)\right] = \lambda^{-2}\frac{\partial^3 F}{\partial z^3} \tag{2.5}$$

and
$$\frac{\partial G}{\partial t} + 2\epsilon\left[G\frac{\partial F}{\partial z} - F\frac{\partial G}{\partial z}\right] = \lambda^{-2}\frac{\partial^2 G}{\partial z^2}, \tag{2.6}$$

together with a third equation which serves only to determine the axial pressure gradient after the velocity components have been found. It is convenient for the subsequent analysis to eliminate the radial pressure gradient $K(t)$ from (2.5), by differentiation with respect to z. We have then, in place of (2.5),

$$\frac{\partial^3 F}{\partial z^2 \partial t} - 2\epsilon\left[F\frac{\partial^3 F}{\partial z^3} + G\frac{\partial G}{\partial z}\right] = \lambda^{-2}\frac{\partial^4 F}{\partial z^4}. \tag{2.7}$$

The boundary conditions (1.1) and (1.3) now take the form

$$F = \partial F/\partial z = 0, \quad G = e^{it} \quad \text{on} \quad z = 0, \tag{2.8}$$
$$F = \partial F/\partial z = 0, \quad G = \mu e^{it} \quad \text{on} \quad z = 1, \tag{2.9}$$

where μ is a constant which takes the values 0 and 1, corresponding to the two cases (1.3*a*) and (1.3*b*) respectively.

A natural approach to the solution of the system (2.6)–(2.9) consists in the expansion of the functions F and G in series

$$F(z,t;\lambda;\epsilon) = \sum_{n=0}^{\infty} \epsilon^n F_n(z,t;\lambda), \quad G(z,t;\lambda;\epsilon) = \sum_{n=0}^{\infty} \epsilon^n G_n(z,t;\lambda). \tag{2.10}$$

This immediately linearizes (2.6) and (2.7), thereby rendering the problem tractable. However, the basis of (2.10) is the hypothesis that the series constitute valid asymptotic representations of F and G in the limit $\epsilon \to 0$, and it has been shown by Rosenblat (1960) that this is in fact not always the case. More precisely, the situation has been found to be as follows. (i) The series are available without qualification when $\lambda = 0(1)$. (ii) When $\lambda \gg 1$, (2.10) provide uniform asymptotic expansions for limited values of z only, namely, within boundary layers adjacent to moving boundaries. (iii) Outside such layers, the first and second terms in the series for the time-independent component of F are in the ratio $1:\lambda$. Consequently, in this region the expansion can be regarded as a valid approximation only when

$$p \equiv \epsilon\lambda \ll 1. \tag{2.11}$$

The solutions when (2.11) holds have been given by Rosenblat (1960). In this paper we therefore concentrate on the alternative case, when

$$p \nless 1. \tag{2.12}$$

In view of the preceding remarks, it will clearly be necessary to use expansions other than (2.10) in appropriate ranges of z, and to attempt to link them by a matching procedure.

It is noteworthy that the parameter p^2 is closely analogous to the Reynolds number R_s, introduced by Stuart (1963) to describe the steadv streaming generated by an oscillating cylinder.

3. Shear layer solutions

It is apparent that when $\lambda \gg 1$ a Stokes shear layer, of thickness order λ^{-1}, will form adjacent to each moving boundary. The boundary-layer solutions are found, as is usual, by an appropriate scaling of variables followed by taking the limit $\lambda \to \infty$.

Introduce boundary-layer variables defined by

$$\eta_l = \lambda z \quad \text{near} \quad z = 0, \quad \eta_u = \lambda(1-z) \quad \text{near} \quad z = 1. \tag{3.1}$$

We then postulate that the velocity components can be described by writing

$$\left.\begin{aligned} G(z,\ldots) &\sim g_l(\eta_l,\ldots)+O(1/\lambda),\\ F(z,\ldots) &\sim \lambda^{-1}f_l(\eta_l,\ldots)+O(1/\lambda^2),\end{aligned}\right\} \quad \text{near } z = 0, \tag{3.2a}$$

$$\left.\begin{aligned} G(z,\ldots) &\sim g_u(\eta_u,\ldots)+O(1/\lambda),\\ F(z,\ldots) &\sim -\lambda^{-1}f_u(\eta_u,\ldots)+O(1/\lambda^2),\end{aligned}\right\} \quad \text{near } z = 1. \tag{3.2b}$$

The effect of these transformations is that the pairs of functions g_l, f_l and g_u, f_u satisfy the *same* equations in their respective arguments. These equations are found by substituting (3.2) into (2.6) and (2.7), and retaining only the leading terms as $\lambda \to \infty$. We obtain, near both boundaries,

$$\frac{\partial g}{\partial t}+2\epsilon\left[g\frac{\partial f}{\partial \eta}-f\frac{\partial g}{\partial \eta}\right] = \frac{\partial^2 g}{\partial \eta^2} \tag{3.3}$$

and

$$\frac{\partial^3 f}{\partial \eta^2 \partial t}-2\epsilon\left[f\frac{\partial^3 f}{\partial \eta^3}+g\frac{\partial g}{\partial \eta}\right] = \frac{\partial^4 f}{\partial \eta^4}. \tag{3.4}$$

The boundary conditions (2.8) and (2.9) are now replaced by

$$f = \partial f/\partial \eta = 0, \quad g = \mu e^{it} \quad \text{on} \quad \eta = 0, \tag{3.5}$$

where $\mu = 1$ on the lower plane, and $\mu = 0$ or 1 on the upper plane, together with the requirement that solutions so obtained should match with appropriate interior solutions, beyond the shear layers, asymptotically as $\eta \to \infty$. This implies the rejection of any exponentially-growing terms which arise.

The functions f and g, as defined in (3.2), depend on the parameter ϵ, and it is now supposed that they admit series expansions in powers of ϵ which are valid representations as $\epsilon \to 0$. We substitute

$$f = \sum_{n=0}^{\infty} \epsilon^n f_n(\eta, t), \quad g = \sum_{n=0}^{\infty} \epsilon^n g_n(\eta, t) \tag{3.6}$$

into (3.3) and (3.4), and equate coefficients of like powers of ϵ.

The first pair of equations so obtained govern the zero-order functions g_0 and f_0. After applying boundary conditions at $\eta = 0$ and matching with the corresponding interior, inviscid solution (which is zero), we find

$$g_0 = \mu e^{it-\sqrt{i}\eta}, \quad f_0 = 0 \tag{3.7}$$

(it being understood that the real part of g_0 is to be taken).

The next pair of equations, order unity in ϵ, gives by a similar procedure that

$$g_1 = 0,$$

while f_1 is seen to be the sum of a time-independent and a second-harmonic component. The former, which is of major interest, we denote by $f_1^{(s)}$. It is found to be

$$f_1^{(s)} = \mu^2 \left\{ \frac{1}{4\sqrt{2}} (\sqrt{2}\eta - 1 + e^{-\sqrt{2}\eta}) + \alpha\eta^2 + \beta\eta^3 \right\}, \tag{3.8}$$

where the constants α and β are to be determined from the matching. The second-harmonic term can easily be obtained, but it is not necessary to state it explicitly.

Reverting now to the functions F and G through (3.1) and (3.2), and denoting the time-independent part of F by $F^{(s)}$, we have thus far,

$$\left. \begin{aligned} G &\sim \{e^{it-\sqrt{i}\eta_l} + O(\epsilon^2)\} + O(\lambda^{-1}), \\ F^{(s)} &\sim \lambda^{-1} \left\{ \epsilon \left[\frac{1}{4\sqrt{2}} (\sqrt{2}\eta_l - 1 + e^{-\sqrt{2}\eta_l}) + \alpha_l \eta_l^2 + \beta_l \eta_l^3 \right] + O(\epsilon^2) \right\} + O(\lambda^{-2}) \end{aligned} \right\} \tag{3.9}$$

near $z = 0$; and

$$\left. \begin{aligned} G &\sim \{\mu e^{it-\sqrt{i}\eta_u} + O(\epsilon^2)\} + O(\lambda^{-1}), \\ F^{(s)} &\sim -\lambda^{-1} \left\{ \epsilon\mu^2 \left[\frac{1}{4\sqrt{2}} (\sqrt{2}\eta_u - 1 + e^{-\sqrt{2}\eta_u}) + \alpha_u \eta_u^2 + \beta_u \eta_u^3 \right] + O(\epsilon^2) \right\} + O(\lambda^{-2}) \end{aligned} \right\} \tag{3.10}$$

near $z = 1$. At the upper boundary we see that G and $F^{(s)} \sim 0$ when $\mu = 0$; this is to be expected, since there is no Stokes layer adjacent to a boundary at rest.

4. The central region: general considerations

The nature of the solutions in the interior can be inferred from the following discussion which, it should be emphasized, is intended as heuristic argument rather than rigorous analysis.

To the order of magnitude under consideration, both the azimuthal velocity and the fluctuating part of the radial velocity are zero. We omit demonstration of this fact here, since it is a common phenomenon in flows of this type. If there-

fore we separate the radial-axial flow function F into steady and fluctuating components, by writing
$$F = F^{(s)}(z) + F^{(f)}(z,t), \tag{4.1}$$
then we have that, approximately,
$$G \sim 0, \quad dF^{(f)}/dz \sim 0 \tag{4.2}$$
in the interior. The only flow component which persists outside the boundary layers is the steady streaming $F^{(s)}$, and the remainder of this work is concerned with its evaluation.

In view of (4.2), the function $F^{(s)}$ satisfies in the interior the equation
$$-2\epsilon F^{(s)}\frac{d^3F^{(s)}}{dz^3} = \lambda^{-2}\frac{d^4F^{(s)}}{dz^4}, \tag{4.3}$$
which is merely the time-independent version of (2.7), with the negligible contributions from G and $F^{(f)}$ discounted. The boundary conditions (2.8) and (2.9) are now replaced by the requirement that $F^{(s)}$ should match asymptotically with the shear-layer solutions (3.9) and (3.10).

In order to achieve this matching, a suitable scaling of both independent and dependent variables in (4.3) is necessary. This scaling has a second function to fulfil, namely, to render the two terms in (4.3) comparable in magnitude. This is because the linearization procedure discussed earlier, which neglects the non-linear convection term on the left of (4.3) in comparison with the viscous term on the right, is not valid when (2.12) holds.

Let the scaled variable be
$$\zeta \equiv az = (a/\lambda)\eta, \tag{4.4}$$
where a is a constant to be determined, and where clearly
$$a/\lambda \ll 1. \tag{4.5}$$
To attain a balance of terms in (4.4) on this scale we must have
$$F^{(s)} = O(a/\epsilon\lambda^2). \tag{4.6}$$

The value of a is to be found by matching with (3.9) and (3.10), which involve unknown constants α and β. We now show that these constants are asymptotically zero as $\lambda \to \infty$.

Consider initially the case $\mu = 0$, when there is a layer at the lower plane only. Suppose $\alpha_l = 0(1)$, and temporarily let $\beta_l = 0$. Then the dominant term in (3.9) at the edge of the layer is, after re-scaling,
$$F^{(s)} \sim \epsilon\lambda^{-1}.\alpha_l.(\lambda/a)^2\zeta_l^2, \tag{4.7}$$
and this is required to match with a one-term expansion of (3.4). Since (4.7) is quadratic in ζ_l, the boundary conditions on this solution of (4.3) are
$$F^{(s)}, \quad \frac{dF^{(s)}}{d\zeta_l} \to 0 \quad \text{as} \quad \zeta_l \to 0; \tag{4.8}$$
while at the upper plane we have
$$F^{(s)} = \frac{dF^{(s)}}{dz} = 0 \quad \text{as} \quad z \to 1. \tag{4.9}$$

Hence the leading-term solution of (4.3) must satisfy the four homogeneous boundary conditions (4.8) and (4.9). It can be shown rigorously that, as might be expected, this system has no solution other than

$$F^{(s)} \equiv 0, \tag{4.10}$$

(see Jones 1968).
From the correspondence between (4.7) and (4.10) it follows that

$$\alpha_l \sim 0.$$

A similar argument can be used to show that $\beta_l \sim 0$, and that, when $\mu = 1$, both α_u and $\beta_u \sim 0$ also.

The leading term in (3.9) at the edge of the layer is therefore in fact

$$F^{(s)} \sim \epsilon\lambda^{-1}\,.\,\tfrac{1}{4}\eta_l = (\epsilon/a)\,.\,\tfrac{1}{4}\zeta_l, \tag{4.11}$$

near $z = 0$ (but similarly near $z = 1$). Comparing (4.11) with (4.6), we see that

$$a = \epsilon\lambda = p. \tag{4.12}$$

Hence the appropriate transformations for (4.3) are

$$\zeta = pz, \quad F^{(s)} = \lambda^{-1}\phi(\zeta), \tag{4.13}$$

and these lead to the equation

$$-2\phi\frac{d^3\phi}{d\zeta^3} = \frac{d^4\phi}{d\zeta^4} \tag{4.14}$$

outside the Stokes boundary layers.

5. The central region: the case $p = O(1)$

On the basis of the foregoing discussion, it is now possible to determine the flow solution in the central region. Strictly, the correct procedure would be to return to the original equations (2.6) and (2.7), and to seek new expansions, based on the scalings derived in §4, for all the components of F and G. The algebra involved, however, is tedious and the details are omitted (but see Jones 1968). The outcome is simply to confirm that the azimuthal and fluctuating radial velocities are asymptotically zero in the central region, and that only the steady radial-axial flow persists. Thus we go direct to (4.3).

For the purposes of the numerical calculations which follow, we replace the more general $p = O(1)$ by the reasonably typical special case

$$p = 1. \tag{5.1}$$

Then $a = 1$, $\zeta = z$ and (4.13) has the form

$$z = z, \quad F^{(s)} = \epsilon\phi(z), \tag{5.2}$$

which lead to the governing equation (4.14) throughout the interior. The boundary conditions on the solution of (4.14) are

$$\phi \to 0, \quad d\phi/dz \to \tfrac{1}{4} \quad \text{as} \quad z \to 0, \tag{5.3}$$

which are a consequence of the matching condition (4.11). At the upper plane we have

$$\phi \to 0, \quad d\phi/dz \to \tfrac{1}{4}\mu \quad \text{as} \quad z \to 1, \tag{5.4}$$

which represent either the original boundary conditions on a stationary plane, or a second matching, depending on whether $\mu = 0$ or 1.

Numerical solutions of the non-linear system (4.14), (5.3) and (5.4) have been obtained using a standard Runge–Kutta–Gill integration procedure on an

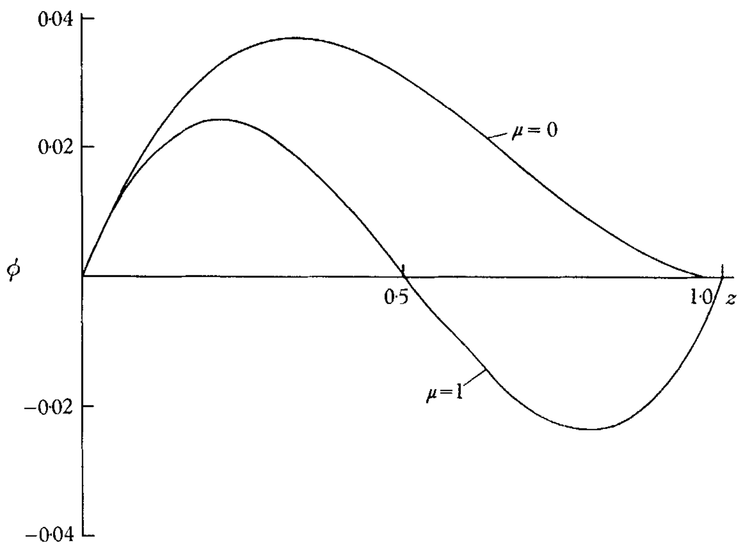

FIGURE 1. Variation of ϕ with z in the case $p = O(1)$.

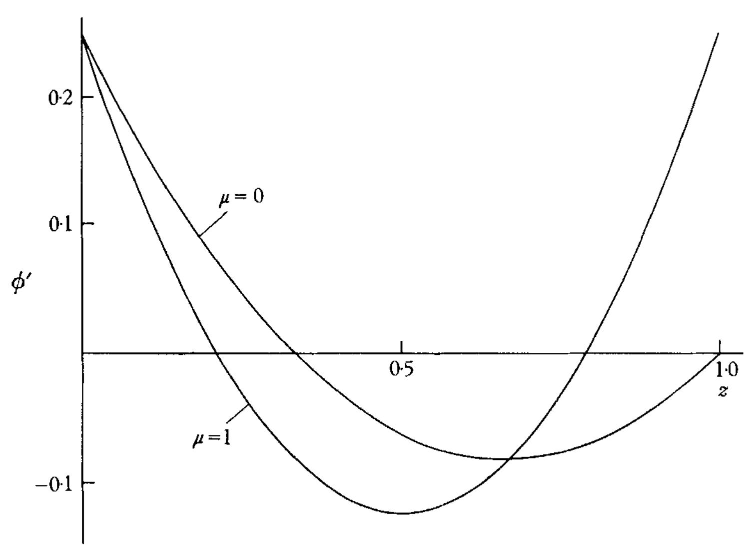

FIGURE 2. Variation of ϕ' with z in the case $p = O(1)$.

IBM 7094 digital computer. The results are shown in figures 1 and 2. Figure 1 illustrates the variation of ϕ (axial velocity) with z for the cases $\mu = 0$, 1, and figure 2 the variation of ϕ' (radial velocity).

When $\mu = 0$, the radial motion is outwards in the part of the central region near the moving boundary, i.e. in the same direction as in the shear layer. Further away, this flow changes direction and becomes radial inflow as far as the plane $z = 1$, balancing the outflow. The axial flow is everywhere directed towards the moving plane.

When $\mu = 1$, there is symmetry of ϕ' about the mid-point $z = \frac{1}{2}$. The radial flow is outward near both boundaries and inward in the centre. The axial flow changes sign at the mid-point.

6. The central region: the case $p \gg 1$

The arguments of §4 continue to apply, but the scaled system (4.13) and (4.14) is now valid only within a distance $O(p^{-1})$ from each boundary. Thus we have the formation of a second boundary layer for the steady flow, of thickness $O(\epsilon^{-1})$ greater than the Stokes layer. The same phenomenon occurs in a semi-infinite fluid (Rosenblat 1959). Equation (4.14) has now to satisfy

$$\phi \to 0, \quad \phi' \to \tfrac{1}{4} \quad \text{as} \quad \zeta \to 0, \tag{6.1}$$

which are the conditions of matching with the Stokes-layer solution. But as $\zeta \to \infty$ we require that the solution of (4.14) should match with an appropriate solution in an interior region, beyond the second boundary layer.

For convenience we shall refer to this latter region as the inner core. In this core we put

$$F^{(s)}(z) = \lambda^{-1}\psi(z), \quad z = z, \tag{6.2}$$

this transformation being justified *a posteriori*. Substituting (6.2) into (4.14), and taking the limit $p \to \infty$, we obtain

$$d^3\psi/dz^3 = 0, \tag{6.3}$$

where ψ in (6.3) is understood to be the leading term in an expansion of (6.2) as $p \to \infty$. Equation (6.3) is seen to be the equation of inviscid flow in the inner core.

For general values of μ, the procedure of matching the solution of (6.3) with the solutions (near each boundary) of (4.14) is rather complicated (see Jones 1968). In the special case $\mu = 1$, however, we can simplify matters by invoking symmetry considerations. Clearly the core flow must be symmetrical about $z = \frac{1}{2}$, so that in place of matching at the upper boundary we impose the boundary conditions

$$\psi = d^2\psi/dz^2 = 0 \quad \text{at} \quad z = \tfrac{1}{2}. \tag{6.4}$$

The solution of (6.3)–(6.4) is

$$\psi = k(\tfrac{1}{2} - z), \tag{6.5}$$

where k is an integration constant. Rewriting this in terms of the variable ζ_l, and retaining only the leading component, we have

$$F^{(s)} \sim \lambda^{-1} . \tfrac{1}{2}k, \quad dF^{(s)}/dz \sim -\lambda^{-1} . k \tag{6.6}$$

as $z \to 0$. These therefore represent the matching conditions for (4.14) as $\zeta_l \to \infty$. In the notation of (4.14) they are

$$\phi(\infty) = \tfrac{1}{2}k, \quad d\phi/d\zeta_l(\infty) = -k/p \sim 0. \tag{6.7}$$

A single integration of (4.14) now gives

$$\phi''' + 2\phi\phi'' - \phi'^2 = 0. \tag{6.8}$$

This equation, with boundary conditions (6.1) and $\phi'(\infty) = 0$, occurs in the case when the fluid is semi-infinite, and has been solved numerically by Benney (1964). Benney's paper contains graphs of ϕ and ϕ', so these are not reproduced here. The constant k, proportional to the inflow at the edge of the second layer, is found to be

$$k = 0{\cdot}530\sqrt{2} \approx 0{\cdot}748. \tag{6.9}$$

Taking Benney's solution in the boundary layer, together with the inviscid solution (6.5), we see that the radial flow component is positive throughout the viscous layer and negative in the core. That is, the outflow in the Stokes and second boundary layers is balanced by inviscid flow in the interior. This of course contrasts with the situation described in § 5, where viscosity is active throughout.

The case $\mu = 0$ can be treated in a similar fashion. In place of the symmetry condition (6.4) we now ask that both radial and axial velocity components should vanish at the upper plane $z = 1$. With this requirement, the solution of (6.3) is

$$\psi = K(1-z)^2. \tag{6.10}$$

Rearranging as before, we find that the infinity condition for (4.14) is, in place of (6.7),

$$\phi(\infty) = K, \quad \phi'(\infty) \sim 0. \tag{6.11}$$

Thus, Benney's solution is again relevant in this layer, and the constant K is found to be

$$K \approx 0{\cdot}374. \tag{6.12}$$

We see that the axial flow is always directed towards the lower plane, but that outside the boundary layer it is quadratic in z, rather than linear, as was the case for $\mu = 1$. The radial flow is, as before, an outflow in the boundary layer and an inflow throughout the rest of the domain.

It may appear surprising that the inviscid equation (6.3) should have a solution which satisfies the boundary conditions at $z = 1$. In fact (6.3) cannot hold in a neighbourhood of the upper plane, since viscosity must again become effective there, and the relevant equation can be shown to be (4.14). Fortuitously, however, the solution (6.10) of equation (6.3) is also a solution of (4.14). This is why we have been able to circumvent the further necessary algebra to deal with this region. The details of this analysis, which involves a new scaling around $z = 1$, are given in Jones (1968).

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